Engineering Notes

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Method for Robust Design of Multivariable Feedback Systems

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Nomenclature

$a_d(\omega)$	= lower bound on the relation between the norms of the disturbance and output vectors
$b(\omega)$	= upper bound on the relation between the norms of the maximal admissible control and measurement noise vectors
$b_d(\omega)$	= bound for disturbance rejection requirement
d	= disturbance vector
$E_M(s)$	= multiplicative perturbation matrix
$F(s)$	= prefilter matrix
$G(s)$	= controller matrix
$L(s)$	= nominal open-loop transfer matrix
n	= measurement noise vector
$P(s)$	= perturbed plant transfer matrix
$P_0(s)$	= nominal plant transfer matrix
r	= input vector
$S(s)$	= sensitivity matrix
$T_0(s)$	= nominal closed-loop transfer matrix
u	= control vector
ω_d	= frequency band where the disturbance exists
ω_n	= frequency band where the measurement noise exists
y	= output vector
$z(\omega)$	= bound for stability robustness
$\bar{\sigma}[A]$	= maximum singular value of A
$\underline{\sigma}[A]$	= minimum singular value of A
$\Vert Y \Vert$	= norm of vector Y, $\sqrt{Y^H Y}$

Introduction

THE problem of maintaining the properties of linear multivariable feedback systems in the face of model uncertainty is crucial for practical designers who need to design such systems according to prescribed specifications. Researchers such as Doyle and Stein,[1] Mannerfelt,[3] and others have formulated robustness criteria that guarantee closed-loop stability and disturbance rejection under parameter variations. In this Note those results are extended, using the singular-value properties, to the development of a design procedure that is based on the diagonalization of the closed-loop transfer matrix and that renders a collection of scalar designs.

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Formulation

Consider the two-degree-of-freedom perturbed linear feedback system shown in Fig. 1, where $P_0(s)$, $P(s)$, and $E_M(s)$ are $k\times m$, $k\times m$, $k\times k$ transfer matrices, respectively, related arbitrarily by the model uncertainty

$$P(s) = [I+E_M(s)]P_0(s) \tag{1}$$

$P(s)$ and $P_0(s)$ are assumed to have fewer zeros than poles and a finite number of unstable poles. $G(s)$ and $F(s)$ are $m\times k$, $k\times k$ transfer matrices, respectively, and are assumed to have no perturbations, fewer zeros than poles, and no unstable poles. r, u, d, y, and n are vectors with compatible dimensions. The input/output relation of the perturbed feedback system is

$$Y(s) = S(s)P(s)G(s)[F(s)R(s) - N(s)] + S(s)D(s) \tag{2}$$

where $S(s)$, the sensitivity matrix, is given by

$$S(s) = [I+P(s)G(s)]^{-1} \tag{3}$$

The design problem is the following: Given $P_0(s)$ with uncertainty, known a priori, in its parameters, which belong to a bounded set, a fixed controller $G(s)$ should be found such that the following requirements are satisfied: 1) stability, 2) specified disturbance rejection, and 3) specified limitation on the control vector. In order to choose such a $G(s)$, sufficient conditions are derived that impose bounds on $T_0(s)$, which is given by

$$T_0(s) = [I+P_0(s)G(s)]^{-1}P_0(s)G(s) \tag{4}$$

and on $L(s)$, which is given by

$$L(s) = P_0(s)G(s) \tag{5}$$

Sufficient Conditions for Design Objectives

Sufficient Conditions for Stability

The perturbed feedback system is stable if the following conditions hold[1]: 1) the nominal feedback system [i.e., with $E_M(s)=0$] is stable, 2) $P(s)$ and $P_0(s)$ have the same number of RHP poles, and 3) the following inequality holds:

$$\bar{\sigma}[T_0(j\omega)] < 1/\bar{\sigma}[E_M(j\omega)] \qquad \forall\omega\geq 0 \tag{6}$$

Sufficient Condition for Disturbance Rejection[2]

In order to satisfy the constraint

$$\max_{\Vert D(j\omega)\Vert=1} \Vert Y(j\omega)\Vert \leq 1/a_d(\omega) \qquad \forall\omega\in\omega_d \tag{7}$$

one has to require that

$$\bar{\sigma}[S(j\omega)] \leq 1/a_d(\omega) \qquad \forall\omega\in\omega_d \tag{8}$$

Using the singular-value properties and Eq. (1), Eq. (8) renders

$$\underline{\sigma}(L(j\omega)) \geq \frac{1+a_d(\omega)}{\underline{\sigma}(I+E_M(j\omega))} \qquad \forall \omega \in \omega_d \tag{9}$$

as a sufficient condition for disturbance rejection.

Sufficient Condition for Bounding the Control[2]

In order to satisfy the constraint

$$\max_{\|N(j\omega)\|=1} \|U(j\omega)\| \leq b(\omega) \qquad \forall \omega \in \omega_n \tag{10}$$

one has to require that

$$\bar{\sigma}[(I+P(j\omega)G(j\omega))^{-1}P(j\omega)G(j\omega)] \leq b(\omega)\bar{\sigma}[P(j\omega)] \qquad \forall \omega \in \omega_n \tag{11}$$

Note that while the left hand side of Eq. (6) is a function of the nominal plant parameters, the left hand side of Eq. (11) is a function of the perturbed plant parameters.

Principles of the Design Method

The dynamic behavior of a fairly well-decoupled multivariable feedback system is mainly dependent upon the diagonal elements of the closed-loop transfer matrix. It is then an interesting question to compare the problems of 1) robustness, 2) disturbance rejection, and 3) minimization of the control vector for different, but not necessarily diagonal, closed-loop transfer matrices that have the same diagonal elements.

The following statement was proved by Dickman and Sivan[2]: A solution for the robustness problem is to choose $G(s)$ such that it diagonalizes $T_0(s)$ [and hence also $L(s)$], if such a $G(s)$ exists. When the uncertainty is not "too" large, namely, when it is admissible to replace $P(s)$ by $P_0(s)$, the proposed solution will simultaneously maximize the disturbance rejection and minimize the control vector. For the 2×2 case, this is also the only solution.

Design Procedure

In view of the preceding, the following design procedure is proposed:

1) Choose the sensors and the maximal admissible control vector such that

$$b(\omega)\min\bar{\sigma}[P(j\omega)] > 1.5 \qquad \forall \omega \in \omega_n \tag{12}$$

Since $T_0(s)$ is chosen to be diagonal it results in

$$\bar{\sigma}[T_0(j\omega)] = \max\{|[T_0(j\omega)]_{ii}|\}, \qquad i=1,\ldots,k \tag{13}$$

and since every $[T_0(s)]_{ii}$ is designed to be fairly well stable, (namely $|[T_0(j\omega)]_{ii}| < 3.5$ dB $\forall \omega \geq 0$), then the fulfillment of Eq. (12) will satisfy Eq. (11) under the preceding assumption. Practically, the left hand side of Eq. (12) should be even greater, since in the presence of uncertainty, the perturbed diagonal element $[T_0(s)]_{ii}$ might be greater than 3.5 dB, $\min\bar{\sigma}[P(j\omega)]$ is computed over the parameter variations.

2) Compute the functions

$$z(\omega) = \min \frac{1}{\bar{\sigma}[E_M(j\omega)]} \qquad \forall \omega \geq 0$$

$$b_d(\omega) = \max \frac{1+a_d(\omega)}{\underline{\sigma}(I+E_M(j\omega))} \forall \omega \in \omega_d,$$

both over the parameter variations, and draw the functions on a Bode plot.

3) The controller $G(s)$ will be chosen such that $L(s)$ is diagonal. Each diagonal element of $L(s)$ should be drawn on a Nichols chart, and while stabilized with the nominal plant parameters, its absolute value should be greater than $b_d(\omega)$ for $\omega \in \omega_d$ in order to satisfy Eq. (9). For each frequency it will be verified (using the Nichols chart) that the absolute value of the compatible diagonal element in $T_0(s)$ is smaller than $z(\omega)$ for $\omega \geq 0$ in order to satisfy Eq. (6).

Such a design is compatible with the common practice of designing multivariable feedback systems in a decoupled form, where each signal loop is so designed that the loop gain is high at low frequency in order to achieve "good" disturbance rejection, and then is decreased in such a manner that it will meet the requirements of stability and minimal bandwidth. The prefilter $F(s)$ is chosen such that the tracking requirements are fulfilled, and does not affect the stability, disturbance rejection, or control vector.

For example, consider the plant

$$P(s) = \begin{bmatrix} \dfrac{1}{s-a_1} & \dfrac{1}{s+1} \\ \dfrac{1}{s+a_2} & \dfrac{1}{s+a_3} \end{bmatrix} \tag{14}$$

where

$$a_1 \in [0.6, \quad 1.4]$$
$$a_2 \in [0.9, \quad 1.1]$$
$$a_3 \in [1.8, \quad 2.2] \tag{15}$$

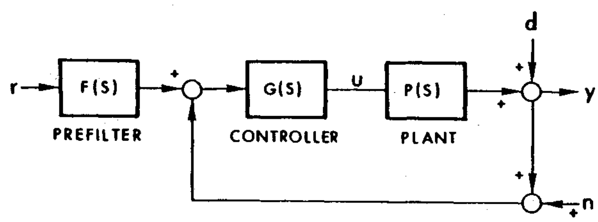

Fig. 1 Perturbed feedback system.

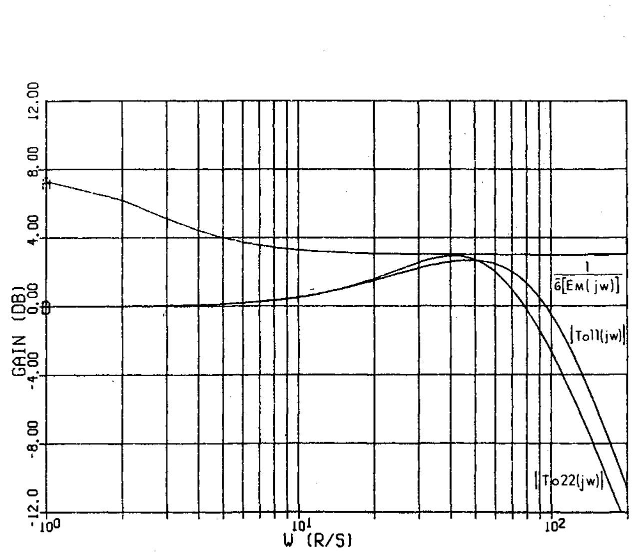

Fig. 2 Maintaining the stability for the perturbed plant.

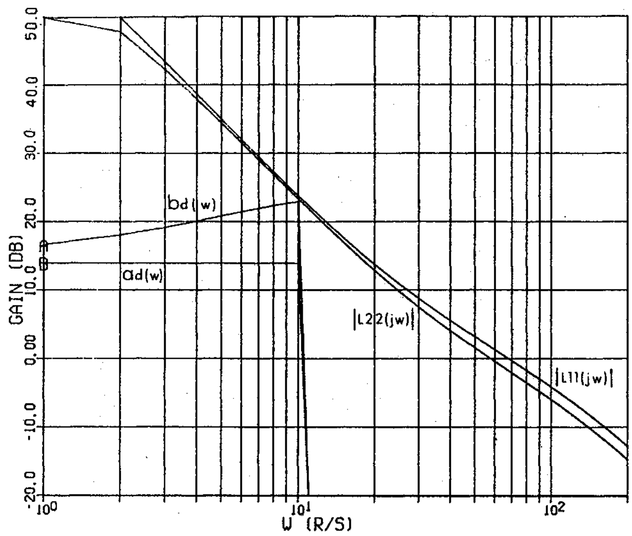

Fig. 3 Disturbance rejection.

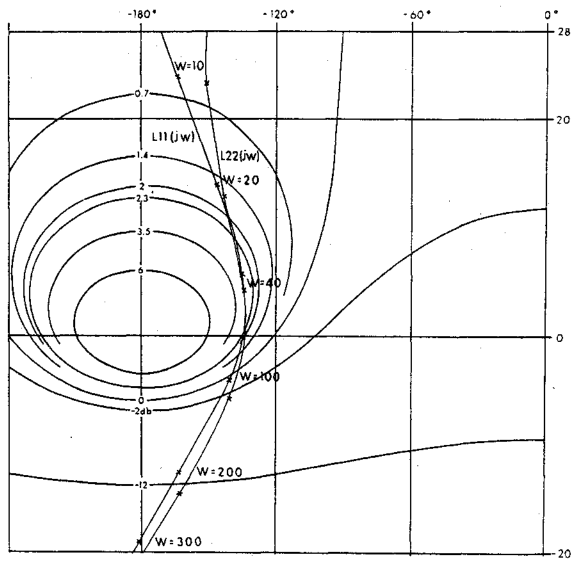

Fig. 4 Nominal stability.

and the nominal plant is given by

$$P_0(s) = \begin{bmatrix} \dfrac{1}{s-1} & \dfrac{1}{s+1} \\ \\ \dfrac{1}{s+1} & \dfrac{1}{s+2} \end{bmatrix} \tag{16}$$

It is required to stabilize the system with $a_d(\omega) = 5$ and $\omega_d \in [0,10]$ r/s.

Design Procedure

1) The sensors and the admissible control vector were chosen such that Eq. (12) was satisfied.

2) $L(s)$ was chosen to be diagonal, and each one of the two diagonal elements was designed separately (see Fig. 4). $|L_{11}(j\omega)|$ and $|L_{22}(j\omega)|$ are both greater than $b_d(\omega)$ in the band ω_d (see Fig. 3) and the compatible elements in $T_0(s)$ are smaller than $z(\omega)$ for $\omega \geq 0$ (see Fig. 2).

$L(s)$, which includes enough far-off poles so that a realizable controller would result, is given by

$$L(s) = \begin{bmatrix} \dfrac{1400(1+s/20)}{s(s-1)(1+s/200)(1+s/500)} & 0 \\ 0 & \dfrac{700(1+s/25)}{s(1+s/2)(1+s/200)(1+s/500)} \end{bmatrix} \tag{17}$$

and the compatible controller is calculated from Eq. (5) and given by

$$G(s) = \begin{bmatrix} \dfrac{466.7(s+1)^2(1+s/20)}{s(1+s/3)(1+s/200)(1+s/500)} & \dfrac{-466.7(s-1)(s+1)(1+s/25)}{s(1+s/3)(1+s/200)(1+s/500)} \\ \\ \dfrac{-933.3(s+1)(1+s/2)(1+s/20)}{s(1+s/3)(1+s/200)(1+s/500)} & \dfrac{466.7(s+1)^2(1+s/25)}{s(1+s/3)(1+s/200)(1+s/500)} \end{bmatrix} \tag{18}$$

Conclusions

The design method that has been described has some limitations that should be emphasized:

1) Since only gain information is considered, the design is conservative, and when there is too much uncertainty in the plant, the bounds that guarantee stability robustness and disturbance rejection cannot be satisfied simultaneously.

2) The choice of the nominal parameters is important for the stability robustness condition, and it seems that choosing the nominal parameters as the algebraic average between the extremal values renders the best results.

3) The bound for stability robustness guarantees stability but not specified gain and phase margins.

4) Since the method requires perfect decoupling of the nominal open-loop transfer matrix, the resulting controller might be complicated.

However, the method enables one to design linear multivariable feedback systems with uncertainty, and considers three of the major problems in feedback systems design.

Acknowledgment

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References

[1]Doyle, J.C. and Stein, G., "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," *IEEE Transactions on Automatic Control*, Vol. AC-26, 1981, pp. 4–16.

[2]Dickman, A. and Sivan, R., "On the Robustness of Multivariable Linear Feedback Systems," *IEEE Transactions on Automatic Control*, Vol. AC-30, 1985, pp. 401–404.

[3]Mannerfelt, C.F., "Robust Control Design with Simplified Models," Ph.D. dissertation, Lund University, 1981.

Testing Matrices for Definiteness and Application Examples That Spawn the Need

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Introduction

IN a recent Note,[1] Prussing correctly observed via counterexamples that the "principal minor test" can lead to erroneous conclusions when used as publicized by several textbooks.[2-4] The version under scrutiny here and in Ref. 1 had been advertised to be a necessary and sufficient test to determine whether a symmetric matrix ($A=A^T$) and/or an Hermetian matrix ($A=\bar{A}^T$, where the vinculum denotes the complex conjugate) is positive semidefinite (c.f., Ref. 48). In fact, the conventional wisdom on this crucial test is incorrect, even though it had been propagated by the abovementioned text-books (similar textbook misrepresentations on this subject[5-9] were further identified in Ref. 10). Additional splendid counterexamples can be constructed using the results of Ref. 13. Since Swamy[10] first alerted the control and estimation community to this error in 1973, a whole new crop of analysts has now sprung up apparently to repeat this same error for which his correction was, perhaps, not spread widely enough.

The fourfold purpose of this Note is:

1) To enthusiastically endorse the corrections of Prussing[1] and Swamy,[10] even though they limited their attention/observations to the effect of testing matrices only for positive *semidefiniteness*.

2) To list additional textbooks that have unfortunately inherited and further propagated this same mistake (offered here as a cautious reminder of which expert opinions to be wary of on this subject).

3) To provide an indication of why this issue is so important in applications (an emphasis that appears to be lacking in Refs. 1 and 10).

4) To describe an easily understood and accessible numerical technique that can serve as the foundation for a computer-based test for positive definiteness/semidefiniteness (as is needed for realistically handling the higher-dimensioned matrices typically encountered in practical industrial applications). The test recommended here differs from the *corrected/augmented* "principal minor test" offered in Refs. 1 and 10 (which is typically applicable only to long-hand calculations for the relatively low-dimensional matrices considered in classroom examples with "nice" numbers).

While the corrected/augmented positive semidefiniteness test offered in Refs. 1 and 10 could also be used as the basis of a numerical test procedure to be implemented on the computer to handle higher-dimensional practical problems, it would require several determinant evaluations that are notoriously computationally unwieldy and probably numerically ill-conditioned. The alternative test offered here is apparently more computationally efficient and numerically well-behaved for this type of application.

Distinguishing the Correct Version

Now that the first purpose of this Note is satisfied, the second purpose is completed by indicating that Ref. 14 (p. 552) and Ref. 15 (pp. 381–382) are, unfortunately, also in error along the same lines as warned against in Refs. 1 and 10. Especially insightful engineering treatments/justification of the correct version of this positive semidefinite test are found, for example, in statements following Eqs. 12.3 in Ref. 44 (pp. 482–483), in Ref. 47 (pp. 384–385), Ref. 11 (p. 307), Ref. 16 (pp. 267–270), Ref. 17 (p. 46), and Ref. 18 (pp. 73–74).

Application Utility

As aptly indicated in Refs. 19–24 with coverage of multitudinous practical navigation and aerospace applications of Kalman filtering (and its mathematical dual of the optimal feedback regulator control of a linear system having the property of minimizing an integral quadratic cost function over either a finite or infinite planning horizon), optimal linear filtering and optimal feedback regulation[25-27] both require the solution of similar Riccati differential equations. For the case of continuous-time linear filtering, the associated Riccati equation is of the form,

$$\dot{P}=F(t)P+PF^T(t)+G(t)Q(t)G^T(t)$$
$$-PH^T(t)R^{-1}(t)H(t)P \tag{1}$$

with symmetric positive definite initial condition

$$P(t_0)=P_0 \tag{2}$$

For Eq. (1), the system matrix $F(t)$, the noise gain matrix $G(t)$, and the process noise covariance intensity matrix $Q(t)$ are defined further below as they occur in the linear system model of Eq. (4). To simplify the discussion, Eqs. (4) and (5) depict a system with time-invariant matrices. As needed in order to employ optimal Kalman filtering, the sensor measurements should be capable of being adequately characterized mathematically as

$$y(t)=H(t)x(t)+v(t) \tag{3}$$

where $H(t)$ is the observation matrix and $v(t)$ the zero mean Gaussian white measurement noise. $R(t)=R^T(t)$, which is the covariance intensity level of the white measurement noise $v(t)$. To be useful in applications, the solution $P(t)$ of the above Riccati equation must be at least positive semidefinite.

As in Ref. 28 (pp. 39–41), consider the time-invariant linear system of the form,

$$\dot{x}(t)=Fx(t)+Gu(t) \tag{4}$$

$$y(t)=Hx(t) \tag{5}$$

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